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The application of the extending symmetry group approach in optical soliton communication

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Abstract
A systematic method which is based on the classical Lie group reduction is used to find the novel exact solution of the nonlinear Schrödinger equation (NLS) with distributed dispersion, nonlinearity and gain or loss. We study the transformations between the standard NLS equation and the NLS equations with distributed dispersion, nonlinearity and gain or loss. Appropriate solitary wave solutions can be applied to discuss soliton propagation in optical fibres, and the amplification and compression of pulses in optical fibre amplifiers.

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1. Introduction

The last four decades witnessed extensive theoretical and experimental studies on optical solitons because of their potential applications in long-distance communication [1, 2]. The invention of high-intensity lasers helped Mollenauer *et al* [3] to verify experimentally the pioneering theoretical work on optical solitons, localized-in-time optical pulse evolution from a nonlinear change in the refractive index of the material, known as the Kerr effect, included by the light intensity distribution. When the combined effects of the intensity-dependent refractive index nonlinearity and the frequency-dependent pulse dispersion exactly compensate each other, the pulse propagates without any change in its shape, that is the optical fibres can support solitons on a balance of nonlinear effects and group velocity dispersion in an anomalous dispersion region. Since then, soliton propagation in optical fibres has been widely studied from the interest of fundamental aspects and the potential for applications theoretically using the nonlinear Schrödinger equation (NLSE). In realistic optical fibres, characteristic parameters of the fibre are not constant but can depend on the location in the optical fibre. In other words, these parameters can have space-coordinate dependence. Therefore, in a real communication system of optical solitons, the transmission of the soliton is described by the NLSE model with varying coefficients

$$\mathrm{i}\psi_z + \beta(z)\psi_{tt} + \gamma(z)|\psi|^2\psi + \mathrm{i}g(z)\psi = 0, \tag{1}$$

where $\beta(z)$ and $\gamma(z)$ denote dispersion coefficients changing with distance along the fibre, and $g(z)$ is a parameter representing the heat-insulating amplification or loss. $\psi(z, t)$ is the complex envelope of the electrical field in a comoving frame. In these optical fibres, the incident soliton is modulated and scattered by the inhomogeneity. We know there are some wealth methods for finding special solutions of a constant coefficient partial differential nonlinear equation such as the inverse scattering transformation (IST), bilinear method, symmetry reductions, Bäcklund and Darboux transformations and so on. Because it is very difficult to solve variable-coefficient nonlinear equations, they were often studied by means of numerical analysis or approximate methods [4–6]. In [7–10], the extended symmetry group approach is introduced to solve the special forms of equation (1). The author of [7] started from the extending symmetry group of the following equation:

$$\mathrm{i}\psi_z + \psi_{tt} + W(z, t, |\psi|)\psi = 0, \tag{2}$$

where W is a function of z, t, ψ, ψ^* and is considered as a dependent variable which is parallel to the variable ψ obtained some exact solutions of the NLS equation whose dispersion changes exponentially along optical fibre. This means that in the extended symmetry group approach equation (2) is regarded as a nonlinear equation even in the case where the potential W does not depend on ψ. This is different from the standard technique in Lie group analysis of PDES, where the symmetry study on equation (2) is restricted to the fact that W is a function of (z, t) and $|\psi|$.

It is well known that for most cases pulse propagation in an optical fibre should be described by the form of equation (1). Therefore it is necessary to consider the group structure of the following equation:

$$\mathrm{i}G(z, t, \psi, \psi^*)\psi_z + V(z, t, \psi, \psi^*)\psi_{tt} + W(z, t, |\psi|)\psi = 0, \tag{3}$$

where G, V and W are considered as three dependent variables which are parallel to the variable ψ. In [7, 8], we only consider the effect of the nonlinear coefficient and therefore only assume that W is a new dependent variable. In this paper we will consider the effect of nonlinear coefficient, dispersion coefficient and gain or loss, and assume that G, V, W are all new dependent variables. Using the group structure of equation (3), we can find the finite transformation between (G, V, W, ψ, z, t) and $(G_1, V_1, W_1, \psi_1, z_1, t_1)$; the functions G_1, V_1, W_1, ψ_1 satisfy the following equation:

$$\mathrm{i}G_1(z_1, t_1, \psi_1, \psi_1^*)\psi_{1_{z_1}} + V_1(z_1, t_1, \psi_1, \psi_1^*)\psi_{1_{t_1t_1}} + W_1(z_1, t_1, |\psi_1|)\psi_1 = 0. \tag{4}$$

The relation between equations (4) and (3) will be given in section 2. In section 2 we will see that the solution $\psi_1(z_1, t_1)$ of equation (4) can be obtained from the solution $\psi(z, t)$ of equation (3) with the help of the finite transformation. If equation (4) can describe some real phenomena of pulse propagation in the optical fibre, and the solution of equation (3) can be obtained easily, we can show some significant results which have potential application in optical fibre communication.

In this paper we obtain some significant equations in the form of equation (4) and their exact solutions by the approach of the expended symmetry Lie group. These solutions include important applications such as the amplification and compression of pulse in optical fibre amplifiers [11, 12]. The importance of the results reported here is two-fold: first, the approach leads to a class of exact solutions to the nonlinear differential equation in a systematic way. In particular, if the original equation satisfied by ψ is solvable, the exact solutions of some special types of variable-coefficient nonlinear equation using finite transformation can be obtained easily. The second and more specific significance of these results lies in their potential application to the design of fibre optical amplifiers, optical pulse compressors and solitary-wave-based communication links.

The organization of this paper is as follows. In section 2, the general symmetry approach is introduced. The exact solutions of some different types of variable-coefficient nonlinear Schrödinger equation are given in section 3. Section 4 is a short summary and discussion.

2. Extended Lie group reduction method and symmetries of the NLS equation

Usually, the Lie group study of equation (3) is restricted to the fact that G, V, W are functions of z, t and $|\psi|$. Here we assume that $G(z, t, \psi, \psi^*)$, $V(z, t, \psi, \psi^*)$ and $W(z, t, |\psi|)$ are three new dependent variables. When G, V and W are considered as three dependent variables, a symmetry of equation (3) is called an extending symmetry to distinguish it from the usual symmetry. The first-order differential operator of the extending symmetry operator and corresponding transformation group of equation (3) are given by

$$\begin{aligned} K = {} & \xi(z, t, \psi, \psi^*)\partial_z + \tau(z, t, \psi, \psi^*)\partial_t + \eta(z, t, \psi, \psi^*)\partial_\psi + \eta^*(z, t, \psi, \psi^*)\partial_{\psi^*} \\ & + g(z, t, \psi, \psi^*)\partial_G + v(z, t, \psi, \psi^*, V)\partial_V + w(z, t, \psi, \psi^*, G, V, W)\partial_W, \end{aligned} \tag{5}$$

$$\begin{aligned} & z_1 = z + \varepsilon\xi(z, t, \psi, \psi^*), \qquad t_1 = t + \varepsilon\tau(z, t, \psi, \psi^*), \qquad \psi_1 = \psi + \varepsilon\eta(z, t, \psi, \psi^*), \\ & \psi_1^* = \psi^* + \varepsilon\eta^*(z, t, \psi, \psi^*), \qquad G_1 = G + \varepsilon g(z, t, \psi, \psi^*, G), \\ & V_1 = V + \varepsilon v(z, t, \psi, \psi^*, V), \qquad W_1 = W + \varepsilon w(z, t, \psi, \psi^*, G, V, W), \end{aligned} \tag{6}$$

where ε is the group parameter. Using the invariance condition of equation (3) under operator (5) and its prolongation,

$$\mathrm{i}g\psi_z + \mathrm{i}G[\eta_z] + v\psi_{tt} + V[\eta_{tt}] + w\psi + \eta W = 0, \tag{7}$$

where $[\eta_z]$, $[\eta_{tt}]$ are the infinitesimalities of ψ_z, ψ_{tt} respectively, and can be expressed as

$$[\eta_z] = (\eta - \xi\psi_z - \tau\psi_t)_z + \xi\psi_{zz} + \tau\psi_{zt}, \tag{8}$$

$$[\eta_{tt}] = (\eta - \xi\psi_z - \tau\psi_t)_{tt} + \xi\psi_{ttz} + \tau\psi_{ttt}, \tag{9}$$

we may obtain the system of determining equations,

$$\begin{aligned} & \xi_\psi = \xi_{\psi^*} = \xi_t = 0, \qquad \tau_\psi = \tau_{\psi^*} = 0, \qquad \eta_{\psi^*} = 0, \qquad \eta_{\psi\psi} = 0, \\ & -\mathrm{i}G\tau_z + 2V\eta_{t\psi} - V\tau_{tt} = 0, \qquad -\mathrm{i}G\xi_z - \mathrm{i}vG/V + \mathrm{i}g + 2\mathrm{i}G\tau_t = 0, \\ & \mathrm{i}G\eta_z - vW\psi/V + V\eta_{tt} - \eta_\psi W\psi + 2\tau_t W\psi + w\psi + W\eta = 0. \end{aligned} \tag{10}$$

Solving equation (10) one can get the infinitesimalities τ, ξ, η, η^* and w. Using these infinitesimalities we can obtain finite transformations of z, t, G, V, W, U. However, we see that it is difficult to solve equation (10). Because we only need to consider some special cases of equation (10) for the most real cases, in the following discussions, we consider some special cases of equation (10) which can be determined easily and possess more important physical significance.

Solving equation (10) under the condition $G = 1$, $g = 0$, $V = V(z, t)$, $v = 0$ leads to the following result,

$$\begin{aligned}
&\xi = \xi(z), \qquad \tau = \frac{1}{2}(\xi' + (\ln V)_z\xi)t + U(z),\\
&\eta = \mathrm{i}\left\{\frac{1}{2V}\left[\frac{1}{4}((\ln V)_z\xi + \xi')_z t^2 + U'(z)t\right] + B(z)\right\}\psi,\\
&\eta^* = -\mathrm{i}\left\{\frac{1}{2V}\left[\frac{1}{4}((\ln V)_z\xi + \xi')_z t^2 + U'(z)t\right] + B(z)\right\}\psi^*\\
&w = \left\{\frac{1}{8V}((\ln V)_z\xi + \xi_z)_z t^2 + \frac{1}{2V}U't + B\right\}_z - \frac{\mathrm{i}}{4}((\ln V)_z\xi + \xi_z)_z - \xi_z W,
\end{aligned} \tag{11}$$

where ξ, U, B are the arbitrary smooth functions of z, $E = B + \mathrm{i}\xi_z + c_1$, with c_1 being an arbitrary constant. The finite transformation of equation (11) under the condition $V = 1, \xi \neq 0$ has the form

$$\begin{aligned}
&f_0(z_1) = \varepsilon + f_0(z), \qquad t_1 = \sqrt{\frac{\xi(z_1)}{2}}\left(f_1(z_1) - f_1(z) + \frac{\sqrt{2}t}{\sqrt{\xi(z)}}\right),\\
&\psi_1(z_1, t_1) = \psi(z, t)\exp(f_2(z_1) - f_2(z)), \qquad \psi_1^*(z_1, t_1) = \psi(z, t)^*\exp(f_2^*(z_1) - f_2^*(z)),\\
&W_1(z_1, t_1, |\psi|) = \frac{2}{\xi(z_1)}\left(W(z, t, |\psi|)\xi(z)/2 + f_3(z_1) - f_3(z)\right),
\end{aligned} \tag{12}$$

where ε is a group parameter,

$$f_0(z_1) = \int^{z_1}\frac{\mathrm{d}\sigma}{\xi(\sigma)}, \qquad f_1(z_1) = \int^{z_1}\frac{\sqrt{2}U(\sigma)}{(\xi(\sigma))^{\frac{3}{2}}}\,\mathrm{d}\sigma,$$

$$\begin{aligned}
f_2(z_1) = \frac{\mathrm{i}}{2}\int^{z_1}\frac{\mathrm{d}\sigma}{\xi(\sigma)}\Bigg(&\frac{1}{8}\xi''(\sigma)\xi(\sigma)\left(f_1(\sigma) - f_1(z) + \frac{\sqrt{2}t}{\sqrt{\xi(z)}}\right)^2\\
&+ U'(\sigma)\sqrt{\xi(\sigma)/2}\left(f_1(\sigma) - f_1(z) + \frac{\sqrt{2}t}{\sqrt{\xi(z)}}\right) + B(\sigma)\Bigg),
\end{aligned}$$

$$\begin{aligned}
f_3(z_1) = \int^{z_1}\Bigg(&-\frac{\mathrm{i}}{8}\xi''(\sigma) + \frac{1}{4}\Bigg(\frac{1}{8}\xi'''(\sigma)\xi(\sigma)\left(f_1(\sigma) - f_1(z) + \frac{\sqrt{2}t}{\sqrt{\xi(z)}}\right)^2\\
&+ U''(\sigma)\sqrt{\xi(\sigma)/2}\left(f_1(\sigma) - f_1(z) + \frac{\sqrt{2}t}{\sqrt{\xi(z)}}\right) + B'(\sigma)\Bigg)\Bigg)\,\mathrm{d}\sigma,
\end{aligned}$$

and we have taken $B = E$ in system (12).

The finite transformation of equation (11) under the condition $V = \exp(-\theta z)$, $\xi = 0$, $U = 0$ can be expressed as

$$\begin{aligned}
&z_1 = z, \qquad t_1 = t, \qquad \psi_1 = \psi\exp(\mathrm{i}B\varepsilon),\\
&\psi_1^* = \psi^*\exp(-\mathrm{i}B\varepsilon), \qquad W_1 = W + B'\varepsilon.
\end{aligned} \tag{13}$$

Taking $G = 1$, $g = 0$ in equation (10) and solving the equation, we obtain

$$\begin{aligned} &\xi = \xi(z), \qquad \tau = \tau(t), \qquad \eta = \left(\frac{\tau_t}{2} + \lambda(z)\right)\psi, \qquad \eta^* = \left(\frac{\tau_t}{2} + \lambda^*(z)\right)\psi^*, \\ &v = -\xi_z V + 2\tau_t V, \qquad w = -\mathrm{i}\lambda_z - \xi_z W - V\tau_{ttt}/2, \end{aligned} \tag{14}$$

where $\xi(z), \lambda(z)$ are the arbitrary functions of z, and $\tau(t)$ is the arbitrary function of time t. The finite transformation of equation (14) for $\tau \neq 0$ and $\xi \neq 0$ possesses the following form:

$$\begin{aligned} &f_0(z_1) = \varepsilon + f_0(z), \quad f_1(t_1) = \varepsilon + f_1(t), \quad f_0(z_1) = \int^{z_1} \frac{\mathrm{d}\sigma}{\xi(\sigma)}, \quad f_1(t_1) = \int^{t_1} \frac{\mathrm{d}\sigma}{\tau(\sigma))}, \\ &\psi_1(z_1, t_1) = \psi(z, t)\sqrt{\frac{\tau(t_1)}{\tau(t)}} \exp\left[\int^{z_1} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma - \int^{z} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right], \\ &\psi_1^*(z_1, t_1) = \psi^*(z, t)\sqrt{\frac{\tau(t_1)}{\tau(t)}} \exp\left[\int^{z_1} \frac{\lambda^*(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma - \int^{z} \frac{\lambda^*(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right], \\ &V_1(z_1, t_1, \psi_1, \psi_1^*) = V(z, t, \psi, \psi^*)\frac{\tau^2(t_1)\xi(z)}{\tau^2(t)\xi(z_1)}, \\ &W_1(z_1, t_1, |\psi_1|^2) = \frac{\xi(z)}{\xi(z_1)} W(z, t, |\psi|^2) - \frac{\mathrm{i}}{\xi(z_1)}(\lambda(z_1) - \lambda(z)) \\ &\qquad - \frac{V(z, t, \psi, \psi^*)\xi(z)}{2\xi(z_1)\tau^2(t)}\left[\int^{t_1} \tau'''(\sigma)\tau(\sigma)\,\mathrm{d}\sigma - \int^{t} \tau'''(\sigma)\tau(\sigma)\,\mathrm{d}\sigma\right]. \end{aligned} \tag{15}$$

The finite transformation of equation (14) for $\tau = 0, \xi = 0$ and $\tau = 0, \xi \neq 0$ can be expressed as

$$\begin{aligned} &z_1 = z, \qquad t_1 = t, \qquad V_1(z_1, t_1, \psi_1, \psi_1^*) = V(z, t, \psi, \psi^*), \\ &W_1(z_1, t_1, |\psi_1|^2) = W(z, t, |\psi|^2) - \mathrm{i}\lambda_z\varepsilon \qquad \psi_1(z_1, t_1) = \psi(z, t)\exp(\lambda(z)\varepsilon), \\ &\psi_1^*(z_1, t_1) = \psi^*(z, t)\exp(\lambda^*(z)\varepsilon) \end{aligned} \tag{16}$$

and

$$\begin{aligned} &t_1 = t, \qquad f_0(z_1) = \varepsilon + f_0(z), \qquad f_0(z_1) = \int^{z_1} \frac{\mathrm{d}\sigma}{\xi(\sigma)}, \\ &V_1(z_1, t_1, \psi_1, \psi_1^*) = V(z, t, \psi, \psi^*)\frac{\xi(z)}{\xi(z_1)}, \\ &W_1(z_1, t_1, |\psi_1|^2) = \frac{\xi(z)}{\xi(z_1)} W(z, t, |\psi|^2) - \frac{\mathrm{i}}{\xi(z_1)}(\lambda(z_1) - \lambda(z)), \\ &\psi_1(z_1, t_1) = \psi(z, t)\exp\left[\int^{z_1} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma - \int^{z} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right], \\ &\psi_1^*(z_1, t_1) = \psi^*(z, t)\exp\left[\int^{z_1} \frac{\lambda^*(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma - \int^{z} \frac{\lambda^*(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right], \end{aligned} \tag{17}$$

respectively.

Choosing $V = 1$, $v = 0$ in equation (10), one can obtain the following result:

$$\begin{aligned} &\xi = \xi(z), \qquad \tau = \tau(t), \qquad \eta = \left(\frac{\tau_t}{2} + C(z)\right)\psi, \qquad \eta^* = \left(\frac{\tau_t}{2} + C_1(z)\right)\psi^*, \\ &g = \xi_z G - 2\tau_t G, \qquad w = -S\mathrm{i}GC_z - \tau_{ttt}/2 - 2\tau_t W. \end{aligned} \tag{18}$$

The finite transformation of equation (18) for $\tau \neq 0, \xi \neq 0$ can be written as

$$f_0(z_1) = \varepsilon + f_0(z), \quad f_1(t_1) = \varepsilon + f_1(t), \quad f_0(z_1) = \int^{z_1} \frac{d\sigma}{\xi(\sigma)}, \quad f_1(t_1) = \int^{t_1} \frac{d\sigma}{\tau(\sigma)}$$

$$\psi_1(z_1, t_1) = \psi(z, t)\sqrt{\frac{\tau(t_1)}{\tau(t)}} \exp\left[\int^{z_1} \frac{C(\sigma)}{\xi(\sigma)} d\sigma - \int^{z} \frac{C(\sigma)}{\xi(\sigma)} d\sigma\right],$$

$$\psi_1^*(z_1, t_1) = \psi^*(z, t)\sqrt{\frac{\tau(t_1)}{\tau(t)}} \exp\left[\int^{z_1} \frac{C^*(\sigma)}{\xi(\sigma)} d\sigma - \int^{z} \frac{C^*(\sigma)}{\xi(\sigma)} d\sigma\right], \tag{19}$$

$$G_1(z_1, t_1, \psi_1, \psi_1^*) = G(z, t, \psi, \psi^*)\frac{\xi(z_1)\tau^2(t)}{\xi(z)\tau^2(t_1)},$$

$$W_1(z_1, t_1, |\psi_1|^2) = \frac{\tau^2(t)}{\tau^2(t_1)} W(z, t, |\psi|^2) + i\frac{G(z, t, \psi, \psi^*)\tau^2(t)}{\xi(z)\tau^2(t_1)}[C(z) - C(z_1)]$$
$$+ \frac{1}{2\tau''(t_1)}\left[\int^{t} \tau'''(\sigma)\tau(\sigma) d\sigma - \int^{t_1} \tau'''(\sigma)\tau(\sigma) d\sigma\right],$$

where we have taken $C^*(z) = C_1(z)$.

3. Several variable-coefficient NLS equations and their solutions

Using the extended symmetry operator (5) and the related finite transformations (12), (13), (15), (16), (17) and (19), we may get some variable-coefficient NLS equations and establish relations among solutions of these different variable-coefficient NLS equations. Only if one of these equations is a constant coefficient equation or a variable-coefficient NLS equation whose solution is well known, the solutions of other variable-coefficient equations can be obtained easily.

With the aid of the finite transformation (12), we get the following variable-coefficient equation,

$$i\psi_{1_{z_1}} + \psi_{1_{t_1 t_1}} + \frac{2\psi_1}{\xi(z_1)}\left(\frac{\xi(z_2)}{2} W(z_2, t_2, \psi_2, \psi_2^*) - f_3(z_2) + f_3(z_1)\right) = 0, \tag{20}$$

where

$$z_2 = f_0^{-1}(f_0(z_1) - \varepsilon), \qquad t_2 = \sqrt{\frac{\xi(z_2)}{2}}\left(\frac{\sqrt{2}t_1}{\sqrt{\xi(z_1)}} + f_1(z_2) - f_1(z_1)\right),$$

$$\psi_2 = \psi_1 \exp(f_2(z_2) - f_2(z_1)), \qquad \psi_2^* = \psi_1^* \exp(f_2^*(z_2) - f_2^*(z_1)),$$

and f_0^{-1} is the inverse function of f_0.

The solution of (20) can be written as

$$\psi_1(z_1, t_1) = \psi(z, t) \exp(f_2(z_1) - f_2(z))|_{t=t_2, z=z_2}, \tag{21}$$

where $\psi(z, t)$ is the solution of the following equation:

$$i\psi_z + \psi_{tt} + W(t, z, |\psi|)\psi = 0. \tag{22}$$

In equation (20) ξ is an arbitrary function of its self-variable and the form of W comes from equation (22) into which t, z, ψ, ψ^* should be substituted by $t_2, z_2, \psi_2, \psi_2^*$. In the same way some other variable-coefficient NLS equations are expressed in the following discussion.

Using the finite transformation (13), the variable-coefficient equation

$$i\psi_{1_{z_1}} + \exp(-\theta z_1)\psi_{1_{t_1 t_1}} + W(t = t_1, z = z_1, \psi = \psi_2, \psi^* = \psi_2^*)\psi_1 = -B'\varepsilon\psi_1 \tag{23}$$

can be obtained. In equation (23), $\psi_2 = \psi_1 \exp(-\mathrm{i}B\varepsilon)$, $\psi_2^* = \psi_1^* \exp(\mathrm{i}B\varepsilon)$. The solution of (23) has the following form:

$$\psi_1(z_1, t_1) = \psi(z, t)\exp(\mathrm{i}B\varepsilon)|_{z=z_1,t=t_1}, \qquad \psi_1^*(z_1, t_1) = \psi^*(z, t)\exp(-\mathrm{i}B\varepsilon)|_{z=z_1,t=t_1}, \tag{24}$$

with $\psi(z, t)$ being the solution of (25):

$$\mathrm{i}\psi_z + \exp(-\theta z)\psi_{tt} + W(t, z, |\psi|)\psi = 0. \tag{25}$$

From the finite transformation (15), we may obtain the variable-coefficient equation

$$\begin{aligned}
\mathrm{i}\psi_{1_{z_1}} &+ V(z_2, t_2, \psi_2, \psi_2^*)\frac{\tau^2(t_1)\xi(z_2)}{\tau^2(t_2)\xi(z_1)}\psi_{1_{t_1t_1}} + \frac{\xi(z_2)}{\xi(z_1)}W(z_2, t_2, \psi_2, \psi_2^*)\psi_1 \\
&= \frac{\mathrm{i}}{\xi(z_1)}(\lambda(z_1) - \lambda(z_2))\psi_1 + \frac{V(z_2, t_2, \psi_2, \psi_2^*)\xi(z_2)}{2\xi(z_1)\tau^2(t_2)} \\
&\quad \times \left[\int^{t_1} \tau'''(\sigma)\tau(\sigma)\,\mathrm{d}\sigma - \int^{t} \tau'''(\sigma)\tau(\sigma)\,\mathrm{d}\sigma|_{t=t_2}\right]\psi_1,
\end{aligned} \tag{26}$$

where

$$z_2 = f_0^{-1}(f_0(z_1) - \varepsilon), \qquad t_2 = f_1^{-1}(f_1(t_1) - \varepsilon),$$

$$\psi_2 = \psi_1(z_1, t_1)\sqrt{\frac{\tau(t_2)}{\tau(t_1)}}\exp\left[\int^{z} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\bigg|_{z=z_2} - \int^{z_1} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right],$$

$$\psi_2^* = \psi_1^*(z_1, t_1)\sqrt{\frac{\tau(t_2)}{\tau(t_1)}}\exp\left[\int^{z} \frac{\lambda^*(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\bigg|_{z=z_2} - \int^{z_1} \frac{\lambda^*(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right].$$

The solution of (26) can be expressed as

$$\psi_1(z_1, t_1) = \psi(z, t)\sqrt{\frac{\tau(t_1)}{\tau(t)}}\exp\left[\int^{z_1} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma - \int^{z} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right]\bigg|_{t=t_2,z=z_2}, \tag{27}$$

where $\psi(z, t)$ is a solution of the following equation:

$$\mathrm{i}\psi_z + V(z, t, \psi, \psi^*)\psi_{tt} + W(z, t, |\psi|)\psi = 0. \tag{28}$$

Using the finite transformation (16), we obtain the variable-coefficient equation,

$$\mathrm{i}\psi_{1_{z_1}} + V(t_2, z_2, \psi_2, \psi_2^*)\psi_{1_{t_1t_1}} + W(t_2, z_2, \psi_2, \psi_2^*)\psi_1 = \mathrm{i}\lambda'\varepsilon\psi_1, \tag{29}$$

where

$$t_2 = t_1, \qquad z_2 = z_1, \qquad \psi_2 = \psi_1 \exp(-\lambda(z_1)\varepsilon), \qquad \psi_2^* = \psi_1^* \exp(-\lambda^*(z_1)\varepsilon).$$

The solution of equation (29) reads

$$\psi_1(z_1, t_1) = \psi(z, t)\exp(\lambda(z_1)\varepsilon)|_{t=t_1,z=z_1}. \tag{30}$$

The function $\psi(z, t)$ in equation (30) is a solution of equation (28).

From the finite transformation (17), we obtain the variable-coefficient equation

$$\mathrm{i}\psi_{1_{z_1}} + V(z_2, t_2, \psi_2, \psi_2^*)\frac{\xi(z_2)}{\xi(z_1)}\psi_{1_{t_1t_1}} + \frac{\xi(z_2)}{\xi(z_1)}W(z_2, t_2, \psi_2, \psi_2^*)\psi_1 = \frac{\mathrm{i}}{\xi(z_1)}(\lambda(z_1) - \lambda(z_2))\psi_1, \tag{31}$$

where

$$
\begin{aligned}
& t = t_1, \qquad z_2 = f_0^{-1}(f_0(z_1) - \varepsilon), \\
& \psi_2 = \psi_1(z_1, t_1) \exp\left[\int^{z} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\bigg|_{z=z_2} - \int^{z_1} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right], \\
& \psi_2^* = \psi_1^*(z_1, t_1) \exp\left[\int^{z} \frac{\lambda^*(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\bigg|_{z=z_2} - \int^{z_1} \frac{\lambda^*(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right].
\end{aligned} \tag{32}
$$

The solution of equation (31) possesses the form

$$
\psi_1(z_1, t_1) = \psi(z, t) \exp\left[\int^{z_1} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma - \int^{z} \frac{\lambda(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right]\bigg|_{t=t_1, z=z_2}, \tag{33}
$$

where $\psi(z, t)$ is a solution of equation (28).

With the help of the finite transformation (19), we can obtain the variable-coefficient equation

$$
\begin{aligned}
& \mathrm{i}\psi_{1_{z_1}} G(z_2, t_2, \psi_2, \psi_2^*) \frac{\xi(z_1)\tau^2(t_2)}{\xi(z_2)\tau^2(t_1)} + \psi_{1_{t_1 t_1}} + \frac{\tau^2(t_2)}{\tau^2(t_1)} W(z_2, t_2, |\psi_2|)\psi_1 \\
& \quad = \left\{ \frac{\mathrm{i}G(z_2, t_2, \psi_2, \psi_2^*)\tau^2(t_2)}{\xi(z_2)\tau^2(t_1)} (C(z_1) - C(z_2)) \right. \\
& \qquad \left. + \frac{1}{2\tau''(t_1)} \left[\int^{t_1} \tau'''(\sigma)\tau(\sigma)\,\mathrm{d}\sigma - \int^{t} \tau'''(\sigma)\tau(\sigma)\right] \mathrm{d}\sigma\bigg|_{t=t_2} \right\} \psi_1,
\end{aligned} \tag{34}
$$

where

$$
\begin{aligned}
& z_2 = f_0^{-1}(f_0(z_1) - \varepsilon), \qquad t_2 = f_1^{-1}(f_1(t_1) - \varepsilon), \\
& \psi_2 = \psi_1(z_1, t_1) \sqrt{\frac{\tau(t_2)}{\tau(t_1)}} \exp\left[\int^{z} \frac{C(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\bigg|_{z=z_2} - \int^{z_1} \frac{C(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right], \\
& \psi_2^* = \psi_1^*(z_1, t_1) \sqrt{\frac{\tau(t_2)}{\tau(t_1)}} \exp\left[\int^{z} \frac{C^*(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\bigg|_{z=z_2} - \int^{z_1} \frac{C^*(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right].
\end{aligned}
$$

The solution of equation (34) reads

$$
\psi_1(z_1, t_1) = \psi(z, t) \sqrt{\frac{\tau(t_1)}{\tau(t)}} \exp\left(\int^{z_1} \frac{C(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma - \int^{z} \frac{C(\sigma)}{\xi(\sigma)}\,\mathrm{d}\sigma\right)\bigg|_{t=t_2, z=z_2} \tag{35}
$$

and the function $\psi(z, t)$ in equation (35) is a solution of the following equation:

$$
\mathrm{i}G\psi_z + \psi_{tt} + W(z, t, |\psi|)\psi = 0. \tag{36}
$$

From the above discussion, one can see that we have obtained six types of space–time-dependent NLS equations (34), (31), (29), (26), (23) and (20). The solutions of these equations can be obtained from the solutions of the corresponding space–time-dependent NLS equations (36), (28), (25) and (22). In equations (34), (31), (29), (26), (23) and (20), ξ, τ, λ and C are all arbitrary functions of their self-variable and G, V, W come from the corresponding functions of equations (36), (28), (25) and (23) into which the self-variable of functions G, V, W should be substituted by t_2, z_2, ψ_2, ψ_2^*. Only if we select arbitrary functions in (36), (31), (28), (26), (23) and (20) appropriately, we can discuss some significant equations in physical application and optical fibre communication. In the following we will provide some concrete examples. Taking $\xi = z^2$, $U = 0$, $B = \mathrm{i}z$, $W = |\psi|^2$ in equation (20), equation (20) becomes

$$\mathrm{i}\psi_{1_{z_1}} + \psi_{1_{t_1t_1}} + \frac{1}{1+\varepsilon z_1}|\psi_1|^2\psi_1 = 0 \tag{37}$$

with the solution

$$\psi_1(z_1, t_1) = \frac{1}{\sqrt{1+\varepsilon z_1}}\psi\left(z = \frac{z_1}{1+\varepsilon z_1}, t = \frac{t_1}{1+\varepsilon z_1}\right)\exp\left(\frac{\mathrm{i}\varepsilon t_1^2}{4(1+\varepsilon z_1)}\right).$$

With the aid of the transformation $z_1 = \frac{\exp(\varepsilon x')-1}{\varepsilon}, \varepsilon = -\theta$, equation (37) becomes a NLS equation which describes soliton propagation in the optical fibre with slowly changing dispersion,

$$\mathrm{i}\psi_{2_{x'}} + \exp(-\theta x')\psi_{2_{t't'}} + |\psi_2|^2\psi_2 = 0 \tag{38}$$

with the solution

$$\psi_2(x', t') = \psi\left(z = \frac{\exp(\theta x') - 1}{\theta}, t = t'\exp(\theta x')\right)\exp\left(\frac{\theta x'}{2} - \frac{\mathrm{i}}{\theta}t'^2 4\exp(\theta x')\right), \tag{39}$$

where ψ is a solution of the standard NLS equation

$$\mathrm{i}\psi_z + \psi_{tt} + |\psi|^2\psi = 0. \tag{40}$$

Selecting $\xi(z_1) = Cz_1, \lambda(z_1) = Cz_1, V = \exp(-\theta z), W = |\psi|^2$ in equation (31), we can obtain a NLS equation which describes soliton propagation in the optical fibre with exponentially changing dispersion, exponentially changing nonlinearity and constant gain or loss,

$$\mathrm{i}\psi_{1_{z_1}} + \alpha\exp(-\alpha\theta z_1)\psi_{1_{t_1t_1}} + \alpha\exp(2z_1(\alpha-1))|\psi_1|^2\psi_1 = \mathrm{i}(1-\alpha)\psi_1, \tag{41}$$

where we have taken $\exp(-C\varepsilon) = \alpha$. The solution of equation (41) reads

$$\psi_1(z_1, t_1) = \psi(z, t)\exp(z_1(1-\alpha))|_{t=t_1, z=z_2=\alpha z_1}, \tag{42}$$

where $\psi(z, t)$ is a solution of equation (25) under the condition $W = |\psi|^2$ and possesses the form of formula (39). Considering the concrete expression of $\psi(z, t)$ we can obtain formula (42) as

$$\psi_1(z_1, t_1) = \psi\left(z = \frac{\exp(\theta x') - 1}{\theta}, t = t'\exp(\theta x')\right)$$
$$\times\exp\left(\frac{\theta x'}{2} - \frac{\mathrm{i}\theta t'^2}{4}\exp(\theta x')\right)\exp(z_1(1-\alpha))|_{t'=t_1, x'=\alpha z_1}. \tag{43}$$

The function $\psi(z, t)$ in equation (43) is a solution of the standard NLS equation (40). This is to say that we have obtained the exact solution of the space–time-dependent NLS equation (41) using the solution of the constant coefficient NLS with the help of the finite transformations (12) and (13).

Taking $V = \exp(-\theta z), W = |\psi|^2, \lambda(z_1) = -\frac{\theta z_1}{2\varepsilon} + \sum_{n=1}^{m}\cos(nz_1)$ in equation (29), one can get the space–time-dependent NLS equation

$$\mathrm{i}\psi_{1_{z_1}} + \exp(-\theta z_1)\psi_{1_{t_1t_1}} + \exp\left(\theta z_1 - 2\varepsilon\sum_{n=1}^{m}\cos(nz_1)\right)|\psi_1|^2\psi_1$$
$$= -\mathrm{i}\left(\frac{\theta}{2} + \varepsilon\sum_{n=1}^{m} n\sin(nz_1)\right)\psi_1. \tag{44}$$

The solution of equation (44) is

$$\psi_1(z_1, t_1) = \psi(z, t)\exp\left(-\frac{\theta z_1}{2} + \varepsilon\sum_{n=1}^{m}\cos(nz_1)\right), \tag{45}$$

where $\psi(z,t)$ is a solution of equation (25) under the condition $W = |\psi|^2$. According to equation (30) the solution of (44) can be written as

$$\psi_1(z_1,t_1) = \psi\left(z = \frac{\exp(\theta x') - 1}{\theta}, t = t'\exp(\theta x')\right)\exp\left(\frac{\theta x'}{2} - \frac{\mathrm{i}\theta t'^2}{4}\exp(\theta x')\right) \times \exp\left(-\frac{\theta z_1}{2} + \varepsilon\sum_{n=1}^{m}\cos(nz_1)\right)\Bigg|_{t'=t_1, x'=z_1}. \tag{46}$$

The function $\psi(z,t)$ in equation (46) is a solution of the standard NLS equation (40).

4. Discussion of the exact solutions of the NLS equation with dispersion, nonlinearity and gain or loss

In this section, we will discuss two examples. The first example is the case of equation (41). Because $\psi(z,t)$ in equation (43) is an arbitrary solution of (40) the exact solution of (41) can be obtained from known particular solution of (40). The single soliton solution of (40) reads

$$\psi = \sqrt{2}l\,\mathrm{sech}(2lkz - lt)\exp(\mathrm{i}((l^2 - k^2)z + kt)), \tag{47}$$

see [6]; substituting equation (47) into equation (43) we can obtain the particular solution of (41)

$$\psi_1(z_1,t_1) = \sqrt{2}l\exp\left(1 - \alpha + \frac{\theta}{2}\alpha\right)z_1)\mathrm{sech}\left(\frac{2kl}{\theta}(\exp(\theta\alpha z_1) - 1) - l\exp(\theta\alpha z_1)t_1\right) \times \exp\left(\mathrm{i}\left(\exp(\theta\alpha z_1)\left(\frac{l^2 - k^2}{\theta} - \frac{\theta}{4}t_1^2 + kt_1\right) - \frac{l^2 - k^2}{\theta}\right)\right). \tag{48}$$

From equation (48), we find the peak amplitude $A(z_1)$, and the pulse width $W(z_1)$ can be expressed as

$$A(z_1) = \sqrt{2}l\exp\left(1 - \alpha + \frac{\theta}{2}\alpha z_1\right), \qquad W(z_1) = \frac{1}{l}\exp(-\theta\alpha z_1). \tag{49}$$

It is remarkable that the width of the sech pulse at $\theta\alpha > 0$ tends to zero when z goes to infinity. This means that the case provides the optimal situation for pulse compression. There are three cases which follow from equation (49) for the peak amplitude $A(z_1)$: (i) $\alpha < \frac{2}{2-\theta}$, $A(z_1)$ grows with increasing z_1, (ii) $\alpha > \frac{2}{2-\theta}$, $A(z_1)$ decreases with increasing z_1 and (iii) $\alpha = \frac{2}{2-\theta}$, $A(z_1) = \mathrm{const}$.

A key consequence of this exact solution for constant loss ($\alpha > 1$) is that the pulse can be compressed to any required degree as $z \longrightarrow \infty$ while maintaining its sech shape and linear chirp in the presence of an exponentially distributed dispersion and nonlinearity parameter.

The solution given by equation (48) at $0 < \alpha < 1$ can also be applied to the problem of pulse compression in optical fibre amplifiers. In this case, the gain $g(z) = -(\alpha - 1)$ is positive if $\theta\alpha > 0$ the width of sech pulse tends to zero as $W(z_1) = \frac{1}{l}\exp(-\theta\alpha z_1)$ when $z \longrightarrow \infty$. According to equation (41) we should use a decreasing nonlinearity parameter together with a decreasing group velocity dispersion parameter in a fibre amplifier to compress the sech pulse to any required width. This is more effective than previous nonlinear amplification of a pulse in the anomalous dispersion regime using the constant parameters which have shown that the pulse tends to break up into a series of pulses due to the combined effects of self-phase modulation and dispersion. Allowing the amplifier to have a distributed gain profile provides other design possibilities for an amplifying pulse compressor. From this solution it also follows

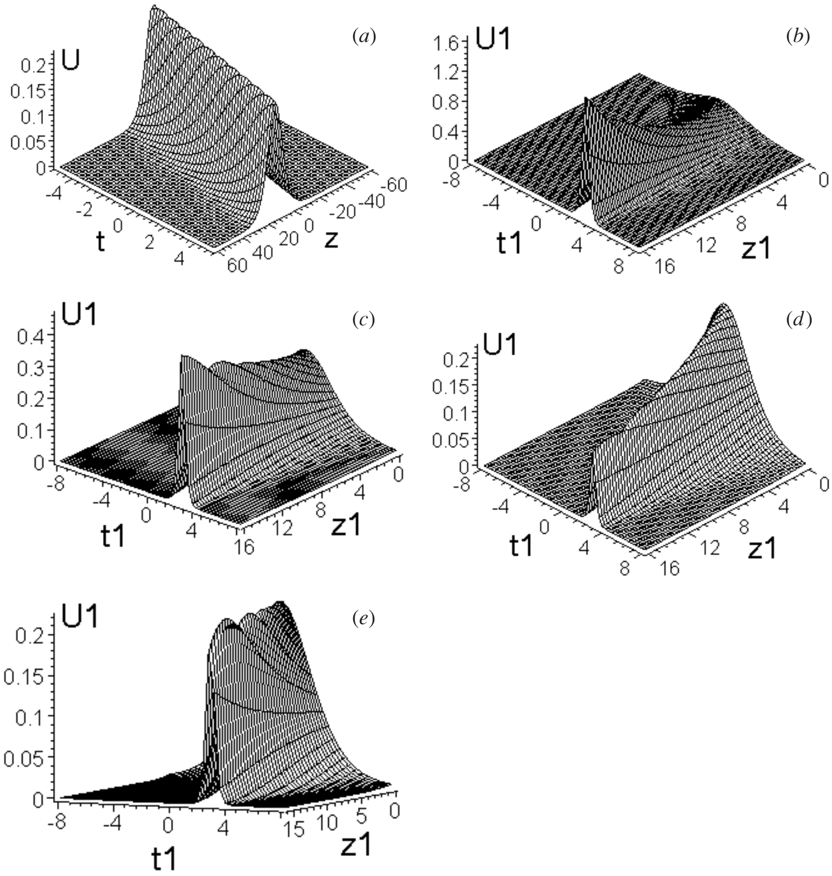

Figure 1. Plots of the exact solution for the NLS equations (40) and (41) for $|\psi|^2 = U$ and $|\psi_1|^2 = U_1$. (*a*) is characterized by (47), and (*b*)–(*e*) are characterized by (48). The parameters in this figure are taken as $l = 1/3, k = 1/5, \theta = 1/8$. The parameters $\alpha = 1, 25/24, 13/12, 16/15$ in (*b*)–(*e*) respectively.

that for $\theta\alpha < 0$ (corresponding to an increasing dispersion parameter), the chirped sech pulse will spread for both amplification and attenuation. The behaviour of the pulse is illustrated in figures 1(*a*)–(*e*) corresponding to (47) and (48) respectively: (*a*) the soliton solution, (*b*) the transformed soliton solution for the case of $\alpha = 1$, (*c*) the transformed soliton solution for case (i), (*d*) the transformed soliton solution for case (ii), (*e*) the transformed soliton solution for case (iii).

From the figure, we see that the solitary wave described by standard NLS equation propagates in optical fibre with its amplitude and pulse width unchanged (see figure 1(*a*)), and the solitary wave described by the NLS equation propagates in optical fibre with slowly changing dispersion, with its amplitude ($U_1 = |\psi_1|^2$) increasing and pulse width decreasing for increasing z_1 (see figure 1(*b*)). We can also see from figure 1(*a*) that the solitary wave described by equation (41) transmits in optical fibre with the change of its amplitude and pulse width the same as that in figure 1(*b*) when $\alpha < \frac{2}{2-\theta}$ (see figure 1(*c*)); however, its amplitude and pulse width decrease for increasing z_1 when $\alpha > \frac{2}{2-\theta}$ (see figure 1(*d*)) and the amplitude remains the same but the pulse width decreases for increasing z_1 when $\alpha = \frac{2}{2-\theta}$ (see

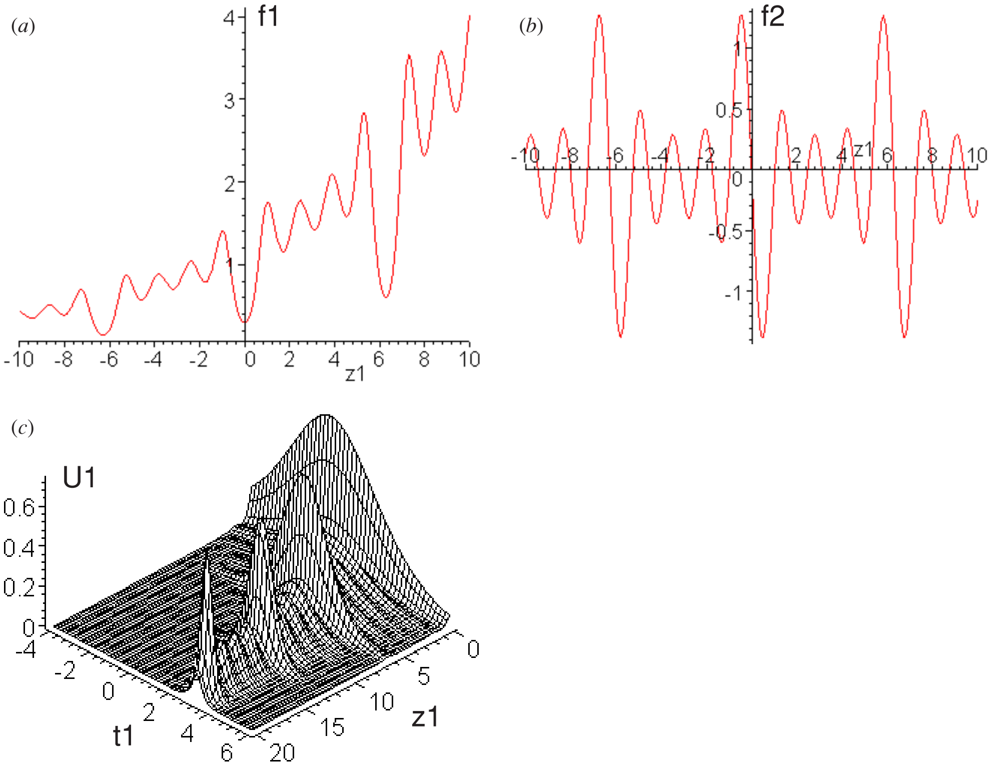

Figure 2. Plots of the exact solution of the NLSE (44) for $|\psi_1|^2 = U_1$. (*a*) and (*b*) are the nonlinear coefficient f_1 and the gain f_2. (*c*) is an exact solution determined by (48). The parameters are taken as $l = 1/3, k = 1/5, \theta = 1/9, m = 4, \varepsilon = 0.15$.

figure 1(*e*)). According to these phenomena we can choose a different parameter α to satisfy different need of soliton application in optical fibre communication.

The second example considered is the case of equation (44). Using equation (47) we can give the solution of equation (44). It can be described by

$$\psi_1(z_1, t_1) = \sqrt{2}l \exp\left(\varepsilon \sum_{n=1}^{m} \cos nz_1\right) \operatorname{sech}\left(\frac{2kl}{\theta}(\exp(\theta z_1) - 1) - l\exp(\theta z_1)t_1\right) \times \exp\left(\mathrm{i}\left(\exp(\theta z_1)\left(\frac{l^2-k^2}{\theta} - \frac{\theta}{4}t_1^2 + kt_1\right) - \frac{l^2-k^2}{\theta}\right)\right), \qquad (50)$$

where m is an arbitrary integer. From (50) we find the peak amplitude $A(z)$ and the pulse width $W(z)$ possess the following form:

$$A(z_1) = \sqrt{2}l \exp\left(\varepsilon \sum_{n=1}^{m} \cos nz_1\right), \qquad W(z_1) = \frac{1}{l}\exp(-\theta z_1). \qquad (51)$$

Equation (51) shows that the width of the sech pulse tends to zero at $\theta > 0$, and the peak amplitude increases and possesses the periodic property when $z_1 \longrightarrow \infty$. That is to say if the nonlinear coefficient and the gain are periodic functions of the fibre distance z_1 we can obtain the pulse compression effect whose peak amplitude possesses periodic property. Figures 2(*a*)–(*c*) are the three plots about the nonlinear coefficient f_1, the gain f_2 and the amplitude $U_1 = |\psi_1|^2$, where $\psi_1(z_1, t_1)$ is described by (50) and $m = 4$: (*a*) the nonlinear coefficient f_1, (*b*) the gain f_2, (*c*) the transformed soliton solution determined by (50).

From the figure, we see that when the nonlinearity coefficient $f_1(z_1)$ and the amplification coefficient $f_2(z_1)$ are periodic functions of distance there is an oscillation solution and the pulse width monotonically decreases with increasing distance. Considering the loss of optical fibre in engineering design, one often uses some equipment to provide periodic amplification. Equation (44) is a good model to describe these phenomena whose soliton propagates in the optical fibre with periodic amplification. Selecting suitable $f_1(z_1)$ and $f_2(z_1)$, one can obtain different internal structure which may be a useful tool in soliton communication.

5. Summary and discussion

In this paper, the NLS equation with distributed coefficients is studied in detail by using the method of the extended symmetry group. With the aid of the finite transformation from the extended symmetry groups, we have set up the relation between the solutions of different variable-coefficient equations.

Because some arbitrary functions of z or t are included in the extended symmetry groups, we can find some variable-coefficient equations with real physical significant by means of some suitable selection of the arbitrary functions. Only if we know one arbitrary solution of all these resulting equations, the corresponding exact solution of the other equations can be obtained easily. Using several finite transformations we may set up the relation between the solution of the equation that we need and the solution of the constant coefficient equation. Therefore one can obtain some exact solutions of the space–time-dependent equation from the solution of the constant coefficient equation.

We have given some kinds variable-coefficient equations with some arbitrary functions and their solutions. Two of these equations which describe the soliton propagating in optical fibre are discussed in detail.

These exact solutions provide powerful theoretical evidence for soliton communication, and can readily be applied to pulse propagation in nonlinear optical fibre amplifiers and optical fibre compressors.

Though we only discuss equations (41) and (44), and give an exact solution for each equation, one can use this method to study more significant equations and obtain more solutions from the solution of constant coefficient equations. We hope this method is useful to study the soliton phenomena in optical fibres and other nonlinear phenomena.

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